Shell Sort

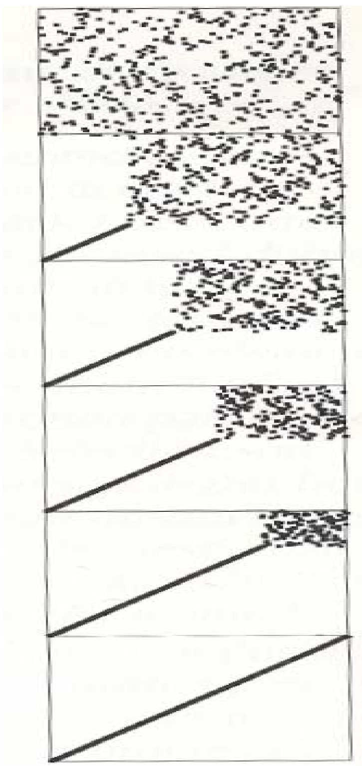
Biostatistics 615/815
Lecture 7

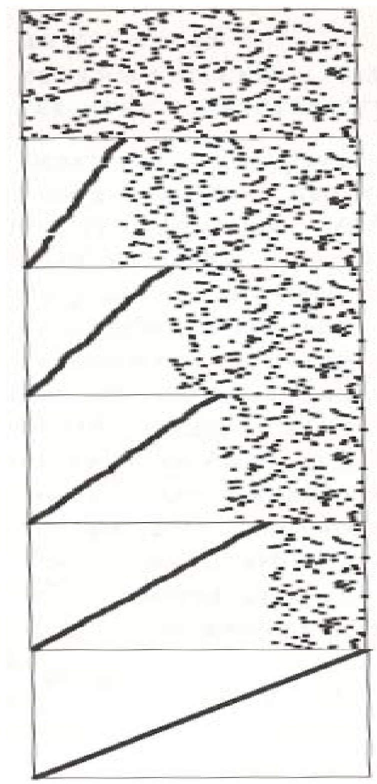
Last Lecture …

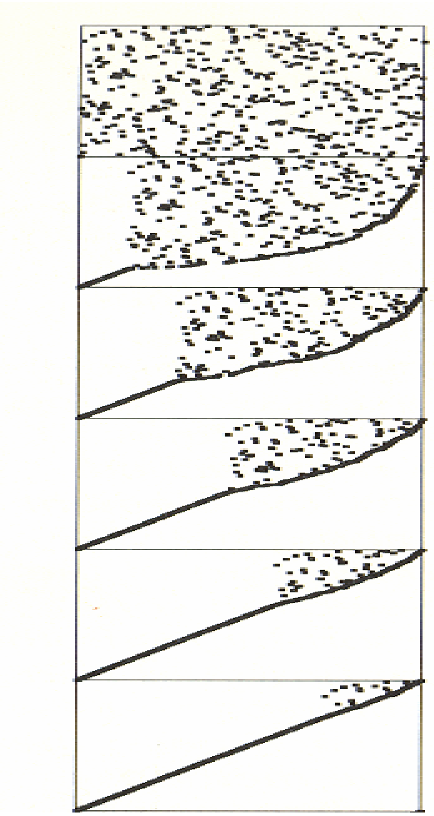
- Properties of Sorting Algorithms
  - Adaptive
  - Stable

- Elementary Sorting Algorithms
  - Selection Sort
  - Insertion Sort
  - Bubble Sort

Selection Insertion Bubble







Recap

- Selection, Insertion, Bubble Sorts
- Can you think of:
  - One property that all of these share?
  - One useful advantage for Selection sort?
  - One useful advantage for Insertion sort?
- Situations where these sorts can be used?

Today …

- Shellsort
  - An algorithm that beats the $O(N^2)$ barrier
  - Suitable performance for general use
- Very popular
  - It is the basis of the default R sort() function
- Tunable algorithm
  - Can use different orderings for comparisons

Shellsort

- Donald L. Shell (1959)
  - *A High-Speed Sorting Procedure*
    Communications of the Association for Computing Machinery **2**:30-32
  - Systems Analyst working at GE
  - Back then, most computers read punch-cards
- Also called:
  - Diminishing increment sort
  - “Comb” sort
  - “Gap” sort

Intuition

- Insertion sort is effective:
  - For small datasets
  - For data that is nearly sorted
- Insertion sort is inefficient when:
  - Elements must move far in array

The Idea …

- Allow elements to move large steps
- Bring elements close to final location
  - First, ensure array is nearly sorted …
  - … then, run insertion sort
- How?
  - Sort interleaved arrays first

Shellsort Recipe

- Decreasing sequence of step sizes *h*
  - Every sequence must end at 1
  - … , 8, 4, 2, 1

- For each h, sort sub-arrays that start at arbitrary element and include every h^{th} element
  - if h = 4
    - Sub-array with elements 1, 5, 9, 13 …
    - Sub-array with elements 2, 6, 10, 14 …
    - Sub-array with elements 3, 7, 11, 15 …
    - Sub-array with elements 4, 8, 12, 16 …

Shellsort Notes

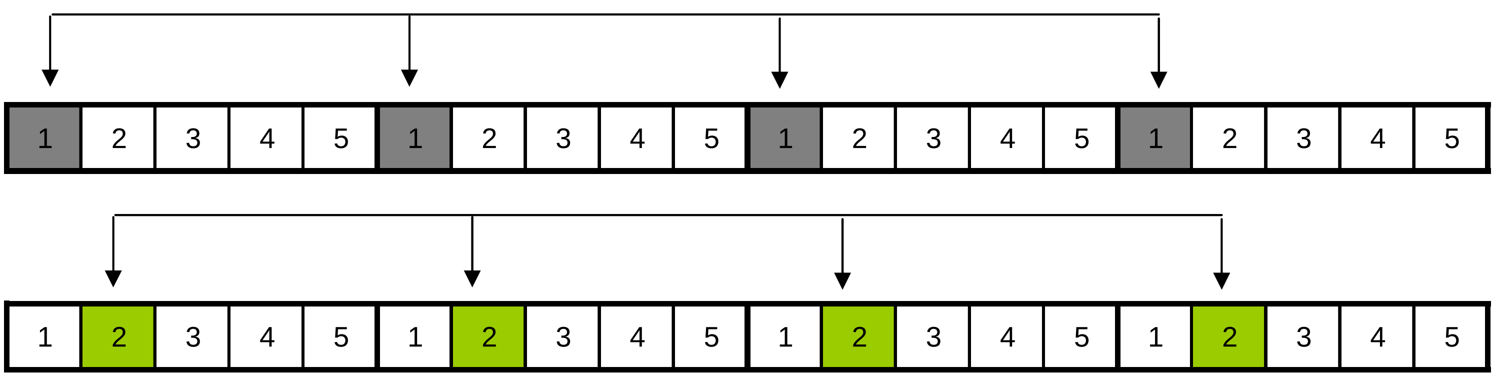
- Any decreasing sequence that ends at 1 will do…
  - The final pass ensures array is sorted

- Different sequences can dramatically increase (or decrease) performance

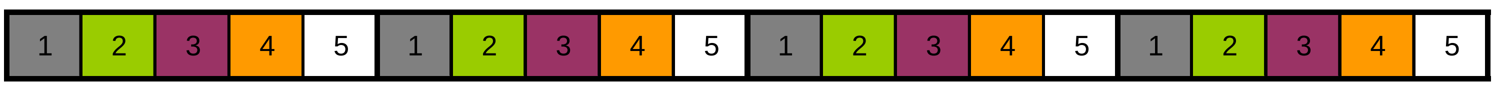
- Code is similar to insertion sort

Sub-arrays when Increment is 5

5-sorting an array



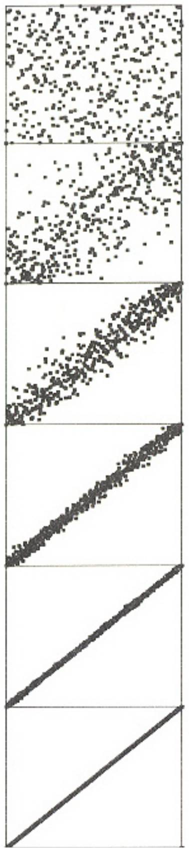
Elements in each subarray color coded



C Code: Shellsort

```
void sort(Item a[], int sequence[], int start, int stop)
   {
   int step, i;

   for (int step = 0; sequence[step] >= 1; step++)
      {
      int inc = sequence[step];

      for (i = start + inc; i <= stop; i++)
         {
         int j = i;
         Item val = a[i];

         while ((j >= start + inc) && val < a[j - inc])
            {
            a[j] = a[j - inc];
            j -= inc;
            }

         a[j] = val;
         }
      }
   }
```

Pictorial Representation



- Array gradually gains order
- Eventually, we approach the ideal case where insertion sort is O(N)

C Code: Using a Shell Sort

```
#include “stdlib.h”
#include “stdio.h”

#define Item int

void sort(Item a[], int sequence[], int start, int stop);

int main(int argc, char * argv[])
   {
   printf(“This program uses shell sort to sort a random array\n\n”);
   printf(“  Parameters: [array-size]\n\n”);

   int size = 100;
   if (argc > 1) size = atoi(argv[1]);

   int sequence[] = { 364, 121, 40, 13, 4,  1, 0};
   int * array = (int *) malloc(sizeof(int) * size);

   srand(123456);
   printf(“Generating %d random elements …\n”, size);
   for (int i = 0; i < size; i++)
       array[i] = rand();

   printf(“Sorting elements …\n”, size);
   sort(array, sequence, 0, size – 1);

   printf(“The sorted array is …\n”);
   for (int i = 0; i < size; i++)
      printf(“%d “, array[i]);
   printf(“\n”);
   free(array);
   }
```

Note on Example Code: Declaring Variables *"Late"*

- Instead of declaring variables immediately after opening a {} block, wait until first use
  - Possibility introduced with C++

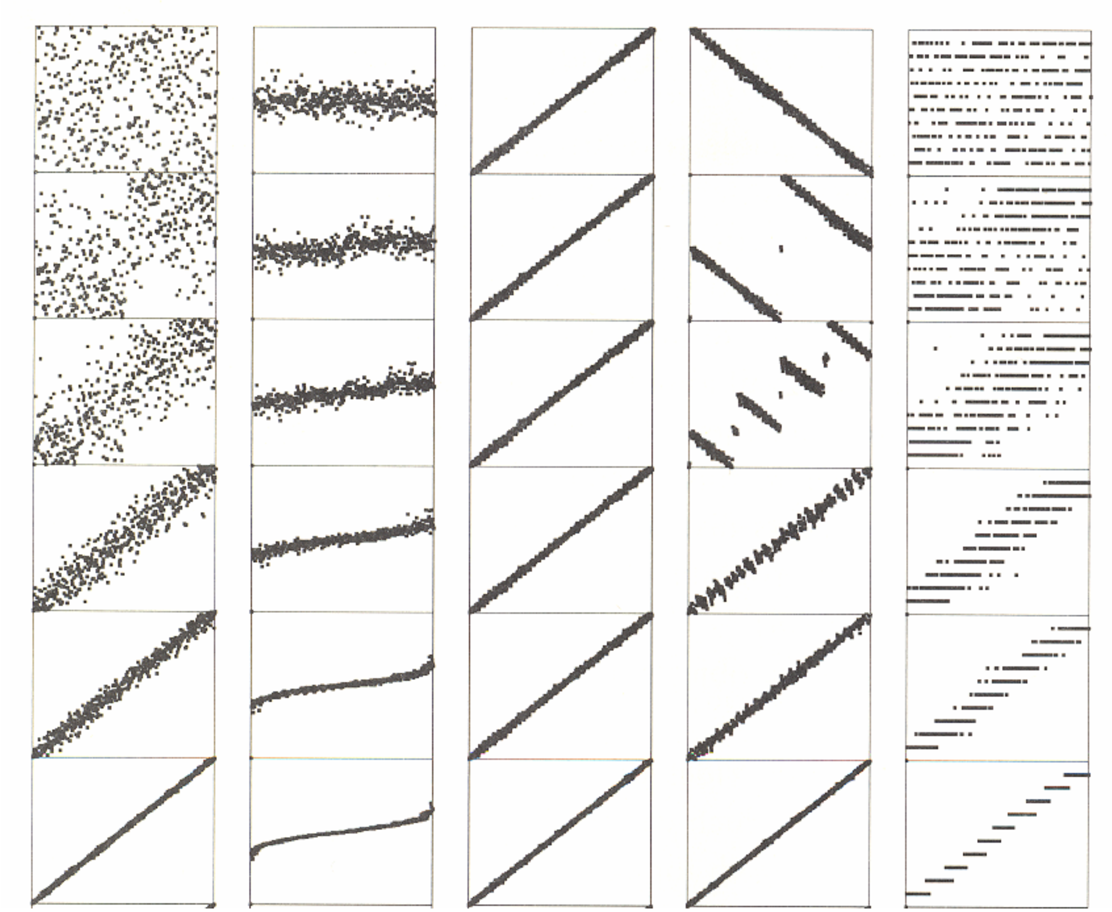
- Supported by most modern C compilers
  - In UNIX, use g++ instead of gcc to compile

Running Time (in seconds)

N	Pow2	Knuth	Merged	Seq1	Seq2
125000	1	0	0	0	0
250000	2	0	0	1	0
500000	6	1	1	0	1
1000000	14	2	2	1	2
2000000	42	5	2	4	3
4000000	118	10	6	7	8

Pow2 – 1, 2, 4, 8, 16 … (2^i)
Knuth – 1, 4, 13, 40, … (3 * *previous* + 1)
Seq1 – 1, 5, 41, 209, … ($4^i - 3 * 2^i + 1$)
Seq2 – 1, 19, 109, 505 … ($9 * 4^i - 9 * 2^i + 1$)
Merged – Alternate between Seq1 and Seq2

Not Sensitive to Input …



Increment Sequences

- Good:
  - Consecutive numbers are relatively prime
  - Increments decrease roughly exponentially
- An example of a bad sequence:
  - 1, 2, 4, 8, 16, 32 …
  - What happens if the largest values are all in odd positions?

Shellsort Properties

- Not very well understood
- For good increment sequences, requires time proportional to
  - $N (\log N)^2$
  - $N^{1.25}$
- We will discuss them briefly …

Definition: *h*-Sorted Array

- An array where taking every h^{th} element (starting anywhere) yields a sorted array
- Corresponds to a set of several* sorted arrays interleaved together
  - * There could be *h* such arrays

Property I

- If we *h*-sort an array that is *k*-ordered…
- Result is an *h*- and *k*- ordered array

- *h*-sort preserves *k*-order!

- Seems tricky to prove, but considering a set of 4 elements as they are sorted in parallel makes things clear…

Property I

- Result of *h*-sorting an array that is *k*-ordered is an *h*- and *k*- ordered array

- Consider 4 elements, in k-ordered array:
  - a[i] <= a[i+k]
  - a[i+h] <= a[i+k+h]

- After h-sorting, a[i] contains minimum and a[i+k+h] contains maximum of all 4

Property II

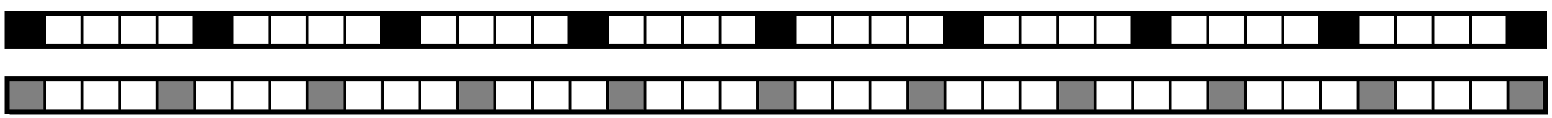
- **If *h* and *k* are relatively prime …**

- Items that are more than *(h-1)(k-1)* steps apart must be in order
  - Possible to step from one to the other using steps size *h* or *k*
  - That is, by stepping through elements known to be in order.

- Insertion sort requires no more *(h-1)(k-1)* comparisons per item to sort array that is *h*- and *k*-sorted
  - Or (h-1)(k-1)/g comparisons to carry a *g*-sort

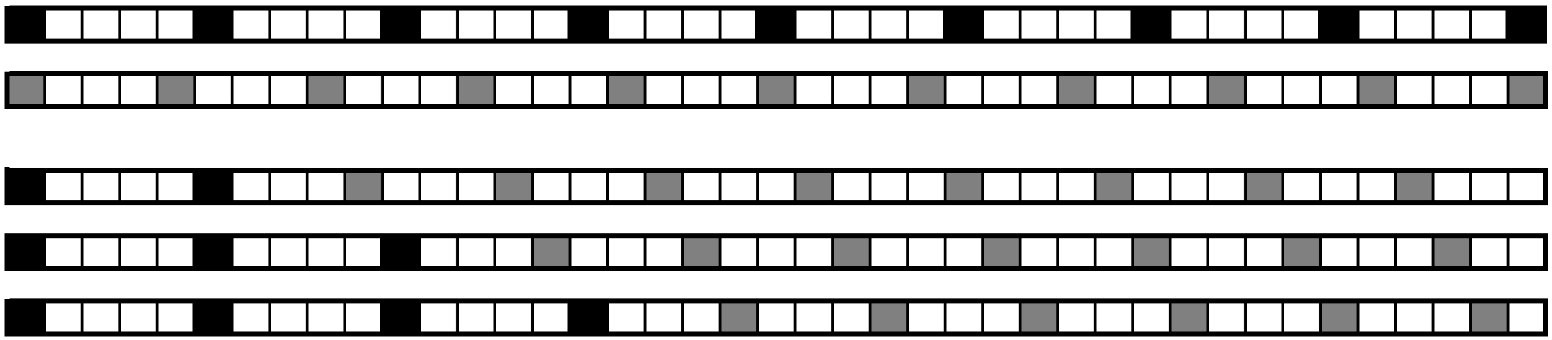
Property II

- Consider *h* and *k* sorted arrays
  - Say h = 4 and k = 5
- Elements that must be in order



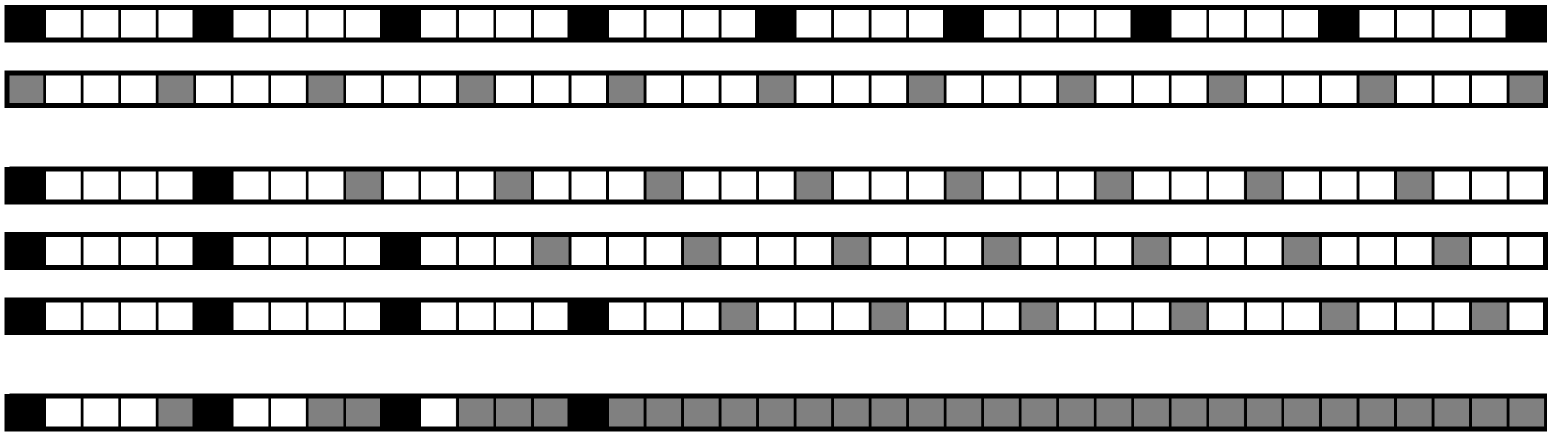
Property II

- Consider *h* and *k* sorted arrays
  - Say h = 4 and k = 5
- More elements that must be in order …



Property II

- Combining the previous series gives the desired property that elements (h-1)(k-1) elements away must be in order



An optimal series?

- Considering the two previous properties…
- A series where every sub-array is known to be 2- and 3- ordered could be sorted with a single round of comparisons
- Is it possible to construct series of increments that ensures this?
  - Before *h*-sorting, ensure 2*h* and 3*h* sort have been done …

Optimal Performance?

- Consider a triangle of increments:
  - Each element is:
    - double the number above to the right
    - three times the number above to the left
  - $< \log_2 N \log_3 N$ increments

```
                    1
                 2     3
              4     6     9
           8    12    18    27
       16    24    36    54    81
    32    48    72   108   162   243
```

Optimal Performance?

- Start from bottom to top, right to left
- After first row, every sub-array is 3-sorted and 2-sorted
  - No more than 1 exchange!
- In total, there are ~ $\log_2 N \log_3 N / 2$ increments
  - About $N (\log N)^2$ performance possible

Today's Summary: Shellsort

- Breaks the N^2 barrier
  - Does not compare all pairs of elements, ever!
- Average and worst-case performance similar
- Difficult to analyze precisely

Reading

- Sedgewick, Chapter 6