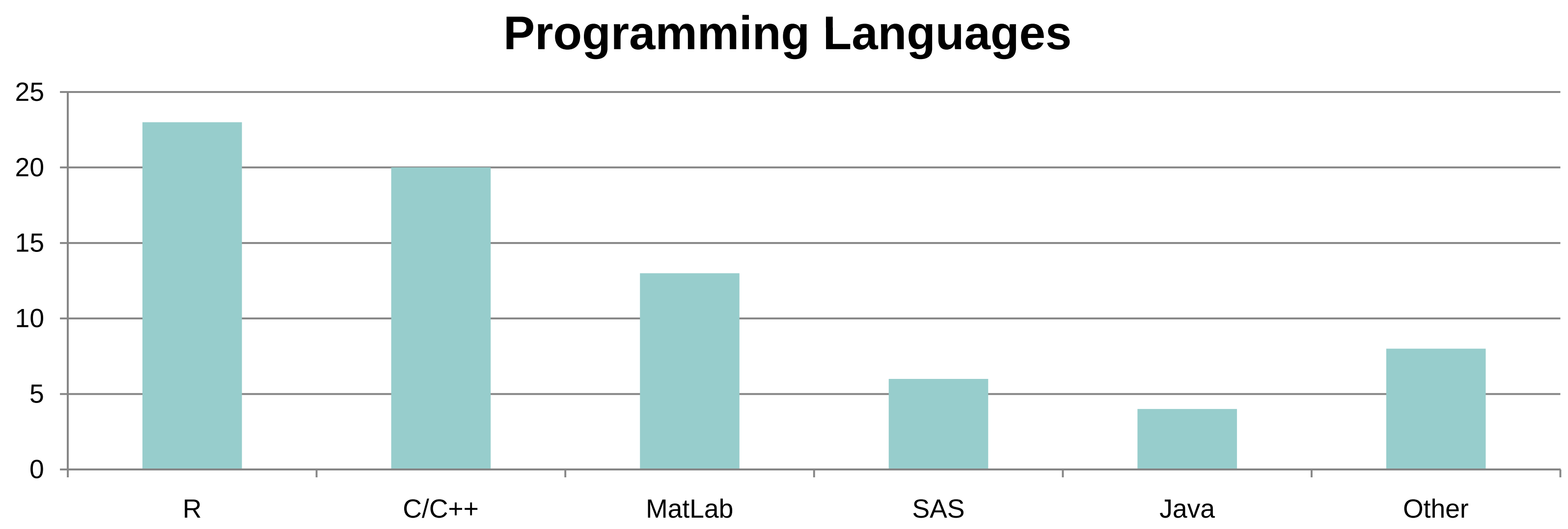
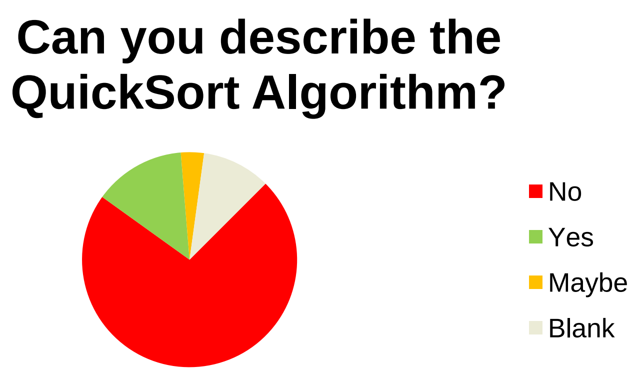
Principles of Algorithm Analysis

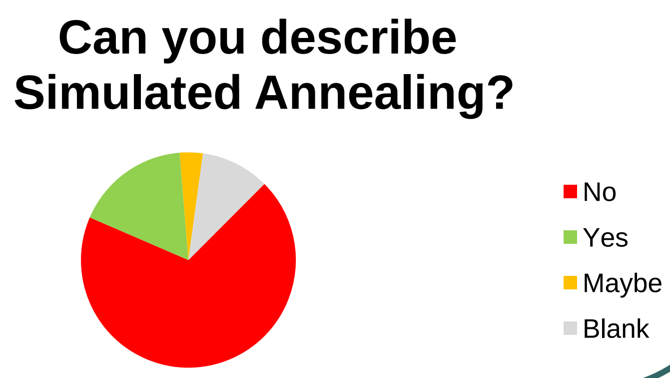
Biostatistics 615/815

Lecture 3

Snapshot of Incoming Class







Homework Notes

- Provide a hard copy including
  - Your Answers
  - Your Code
- Write specific answer to each question
  - Supported by table or graph if appropriate
- Source code
  - Indented and commented, if appropriate

Office Hours

- Wednesdays
  - 1:30 – 4:00 pm
  - SPH Tower, M4614

- Alternatively, e-mail me at:
  - goncalo@umich.edu

Last Week
An Introduction to C

- Strongly typed language
  - Variable and function types set explicitly
- Functional language
  - Programs are a collection of functions
- Compiling and debugging C programs
  - Setup a basic projects
  - Review compile errors and warnings
  - Step through code line by line
  - Set breakpoints

Today

- Strategies for comparing algorithms
- Common relationships between algorithm complexity and input data
- Compare two simple search algorithms

Objectives

- Framework for
  - Empirical Testing
  - Theoretical Analysis
- Highlight performance characteristics of algorithms

Specific Questions

- Compare two algorithms for one task
- Predict performance in a new environment
  - If we had a computer that was 10x faster and could store 10x more data, how would approach perform?
- Set values of algorithm parameters

Two Common Mistakes

- Ignore performance of algorithm
  - Shun faster algorithms to avoid complexity in program
  - Waiting for “simple” but inefficient algorithms to run, when efficient alternatives of modest complexity exist
- Too much weight on performance of algorithm
  - Improving program that is already very fast not worth it
  - Time spent tinkering with code is useful

Empirical analysis

- Given two algorithms … which is better?
- Run both
  - Say, algorithm A takes 3 seconds
  - Say, algorithm B takes 30 seconds
- Empirical studies may not always be practical
  - Some algorithms may take too long to run!
  - Other algorithms may take too long to code…

Choices of Input Data

- Actual data
  - Measures performance in use
- Random data
  - Generic approach, may not be representative
- Perverse data
  - Attempt worst case analysis

Limitations of Empirical Analysis

- Quality of implementation
  - Is our favored implementation coded more carefully than another?
- Extraneous factors
  - Compiler
  - Machine
  - Computer system

Limitations of Empirical Analysis

- Requires a working program
- Theoretical analysis is an alternative
  - Estimate potential gains
- Predict effectiveness relative to new algorithms or computers (that may not yet exist)

Theoretical Analysis

- Predict performance of algorithm based on theoretical properties
- “Independent” of actual implementation
- Several constructs occur frequently in algorithm analysis

Limitations of Theoretical Analysis

- Efficiency can depend on compiler
- Efficiency may fluctuate with input data
- Some algorithms are not well understood

The idea...

- Given a code fragment

```
#Find parent of node i
i = a[i];
```

- Consider how many times it is executed
- But not how long each execution takes

Two typical analyses

- Average-case for random input
- Worst-case
- Are these representative of real world problems?
  - Check with empirical predictions…

The Primary Parameter *N*

- Examples
  - Number of parameters to likelihood function
  - Number of items in dataset to be processed
  - Number of characters in a string
  - Size of file to be sorted
  - Some other abstract measure of problem size
- With multiple inputs, focus on one at a time, while holding the others constant

Running time as a function of *N*

f(N)	Description	Running time when N doubles…
1	*constant*	-
$\log N$	*logarithmic*	constant increase
N	*linear*	doubles
$N \log N$	*log-linear*	more than doubles
N^2	*quadratic*	increases fourfold
N^3	*cubic*	increases eightfold
2^N	*exponential*	running time squares

Running time as a function of *N*

- Multiple terms may be involved
  - e.g. $N + N \log N$
- Typically, we ignore
  - Smaller terms
  - Constant coefficient
  - Focus on inner loop
- In rare cases, smaller terms and constant coefficient will be important

Time to Solve Large Problem

operations per second	Problem Size N = 1,000,000		
	N	N log N	N^2
10^6	seconds	minutes	months
10^9	instant	instant	hours
10^{12}	instant	instant	seconds

Time to Solve Huge Problem

operations per second	Problem Size N = 1,000,000,000		
	N	N log N	N^2
10^6	hours	days	never
10^9	seconds	minutes	centuries
10^{12}	instant	instant	months

Application

- Analysis of two search algorithms
- Each algorithm:
  - Considers a set of items stored in an array
  - Searches through items to decide whether a particular value occurs

Sequential Search

```
int search(int a[], int value, int start, int stop)
   {
   // Variable declarations
   int i;

   // Search through each item
   for (i = start; i <= stop; i++)
       if (value == a[i])
              return i;

   // Search failed
   return -1;
   }
```

Sequential Search Properties

- Algorithm:
  - Look through array sequentially, until we find a match
- Average cost
  - If match found: N/2
  - If match not found: N
- Actual cost depends on fraction of successful searches

Better Sequential Search

- If items are sorted…
- Stop unsuccessful search early, when we reach item with higher value
  - Cost for unsuccessful searches is now N/2
- Overall, algorithm is still O(N)

Binary Search

```
int search(int a[], int value, int start, int stop)
   {
   while (stop >= start)
        {
        // Find midpoint
        int mid = (start + stop) / 2;

        // Compare midpoint to value
        if (value == a[mid])
              return mid;

        // Reduce input in half !...
        if (value > a[mid])
           { start = mid + 1; }
        else
           { stop = mid - 1; }
        }

   // Search failed
   return -1;
   }
```

Binary Search Properties

- Algorithm:
  - Halve number of items to consider with each comparison
- Worst-case cost
  - Maximum cost is never greater than $\log_2 N$
- Much better than sequential search, but even better methods exist!

Sequential vs. Binary Search

	M = 1,000		M = 10,000		M = 100,000	
N	S	B	S	B	S	B
125	1	1	13	2	130	20
250	3	0	25	2	251	22
500	5	0	49	3	492	23
1250	13	0	128	3	1276	25
2500	26	1	267	3	*	28

Timings in seconds, for M searches in table of N elements

Big-Oh Notation

- Algorithm is *O(N)* or *O(N log N)*
  - Common statement
  - What does it mean?
- Summarizes performance for large N
- Focuses on leading terms of expression describing running time

Big-Oh Notation

- Consider function $g(N)$
- It is said to be $O(f(N))$
- If there exist c_0 and N_0 such that:
  - $N > N_0$ implies $c_0 f(N) > g(N)$

From N to Running Time…

- Common relationships
  - N^2
  - $\log N$
  - $N \log N$
  - N

- Describe examples of how these arise
- Cost of running program is C_N

$O(N^2)$

- Loop through input successively, eliminate one item at a time

$$
\begin{aligned}
C_N &= C_{N-1} + N \qquad \text{for } N \geq 2,\ C_1 = 1 \\
&= C_{N-2} + (N-1) + N \\
&\ \ \ldots \\
&= 1 + 2 + \ldots + (N-1) + N \\
&= \frac{N(N+1)}{2}
\end{aligned}
$$

O(log N)

- Recursive program, halves input in one step

$$
\begin{aligned}
C_{2^n} &= C_{2^{n-1}} + 1 \qquad \text{for } N \geq 2,\ C_1 = 1 \\
&= C_{2^{n-2}} + 1 + 1 \\
&= C_{2^{n-3}} + 3 \\
&\ \ldots \\
&= C_{2^0} + n \\
&= n + 1 \\
N &= 2^n
\end{aligned}
$$

O(N log N)

- Recursive program, processes each item, splits input into two halves, examines each one…

$$C_N = 2C_{N/2} + N \qquad \text{for } N \geq 2,\ C_1 = 0$$

$$C_{2^n} = 2C_{2^{n-1}} + 2^n$$

$$\frac{C_{2^n}}{2^n} = \frac{2C_{2^{n-1}} + 2^n}{2^n}$$

$$= \frac{C_{2^{n-1}}}{2^{n-1}} + 1$$

$$= \frac{C_{2^{n-2}}}{2^{n-2}} + 1 + 1$$

$$\ldots$$

$$= n$$

O(2N)

- Halves input, must examine each item…

$$
\begin{aligned}
C_N &= C_{N/2} + N \qquad \text{for } N \geq 2,\ C_1 = 1 \\
&= N + \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \ldots \\
&\approx 2N
\end{aligned}
$$

Summary

- Outline principles for analysis of algorithms
- Introduced some common relationships between *N* and running time
- Described two simple search algorithms

Further Reading

- Read chapter 2 of Sedgewick

Tip of the Day: Defensive Programming

- Document code and programs
  - Indicate intended purpose
  - Specify required inputs
  - Always indicate author

- Check for error conditions