Merge Sort

Biostatistics 615/815
Lecture 8

Notes on Problem Set 2

- Union Find algorithms
- Dynamic Programming

- Results were very positive!

- You should be gradually becoming comfortable compiling, debugging and executing C code

Question 1

- How many random pairs of connections are required to connect 1,000 objects?
  - Answer: ~3,740

- Useful notes:
  - Number of non-redundant links to controls loop
  - Repeat simulation to get a better estimates

Question 2

- Path lengths in the saturated tree…
  - ~1.8 nodes on average
  - ~5 nodes for maximum path
- Random data is far from worst case
  - Worst case would be paths of $log_2 N$ (10) nodes
- Path lengths can be calculated using `weights[]`

Question 3

- Using top-down dynamic programming, evaluate the beta-binomial distribution
  - Like other recursive functions, this one can be very costly to evaluate for non-trivial cases
- Did you contrast results with non-dynamic programming solution?

Last Lecture: Quick Sort

- Choose a partitioning element …
- Organize array such that:
  - All elements to the right are greater
  - All elements to the left are smaller
- Sort right and left sub-arrays independently

Quick Sort Summary

- Divide and Conquer Algorithm
  - Recursive calls can be “hidden”
- Optimizations
  - Choice of median
  - Threshold for brute-force methods
  - Limiting depth of recursion
- Do you think quick sort is a stable sort?

C Code: QuickSort

```c
void quicksort(Item a[], int start, int stop)
  {
  int i;

  if (stop <= start) return;

  i = partition(a, start, stop);
  quicksort(a, start, i - 1);
  quicksort(a, i + 1, stop);
  }
```

C Code: Partitioning

```
int partition(Item a[], int start, int stop)
   {
   int up = start, down = stop – 1, part = a[stop];

   if (stop <= start) return start;

   while (true)
        {
        while (isLess(a[up], part))
             up++;
        while (isLess(part, a[down]) && (up < down))
           down--;

        if (up >= down) break;
        Exchange(a[up], a[down]);
        up++; down--;
        }

   Exchange(a[up], a[stop]);
   return up;
   }
```

Non-Recursive Quick Sort

```
void quicksort(Item a[], int start, int stop)
   {
   int i, s = 0, stack[64];

   stack[s++] = start;
   stack[s++] = stop;

   while (s > 0)
        {
        stop = stack[--s];
        start = stack[--s];
        if (start >= stop) continue;

        i = partition(a, start, stop);
        if (i – start > stop – i)
             {
             stack[s++] = start; stack[s++] = i - 1;
             stack[s++] = i + 1; stack[s++] = stop;
             }
        else {
             stack[s++] = i + 1; stack[s++] = stop;
             stack[s++] = start; stack[s++] = i - 1;
             }
        }
   }
```

Selection

- Problem of finding the k^{th} smallest value in an array
- Simple solution would involve sorting the array
  - Time proportional to *N log N* with Quick Sort
- Possible to improve by taking into account that only one element must fall into place
  - Time proportional to *N*

C Code: Selection

```
// Places kth smallest element in the kth position
// within array. Could move other elements.
void select(Item * a, int start, int stop, int k)
  {
  int i;

  if (start <= stop) return;

  i = partition(a, start, stop);

  if (i > k) select(a, start, i - 1);
  if (i < k) select(a, i + 1, stop);
  }
```

Merge Sort

- Divide-And-Conquer Algorithm
  - Divides a file in two halves
  - Merges sorted halves
- The “opposite” of quick sort
- Requires additional storage

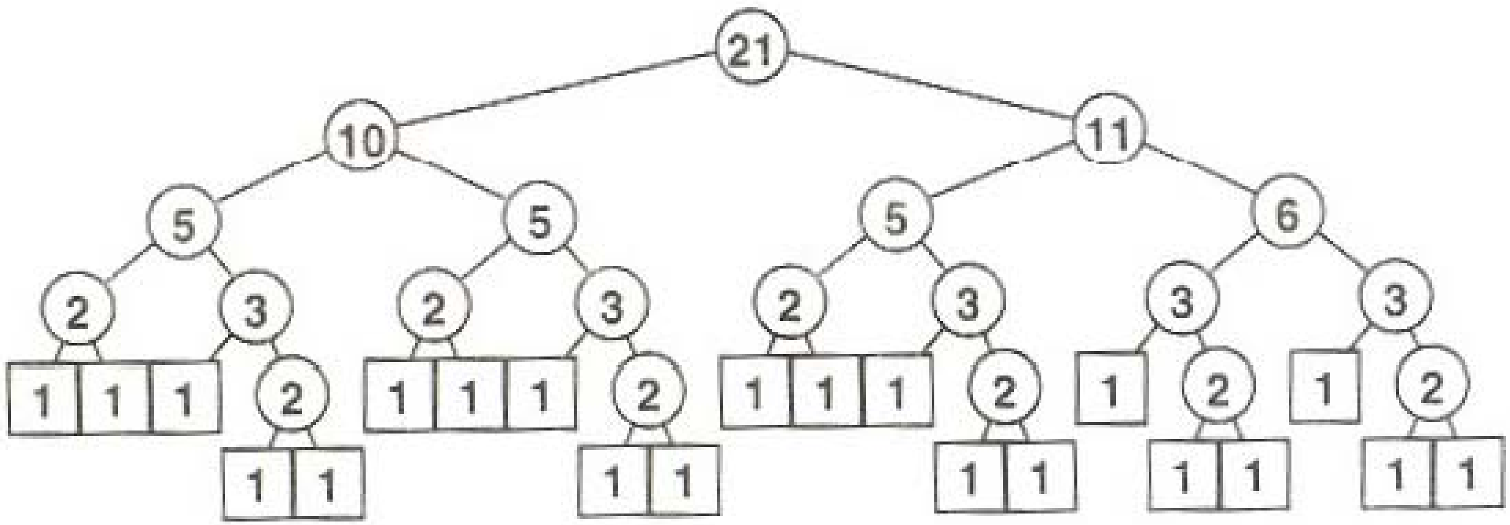
C Code: Merge Sort

```
void mergesort(Item a[], int start, int stop)
  {
  int m = (start + stop)/2;

  if (stop <= start) return;

  mergesort(a, start, m);
  mergesort(a, m + 1, stop);
  merge(a, start, m, stop);
  }
```

Merge Pattern N = 21



Merging Sorted Arrays

- Consider two arrays
- Assume they are both in order
- Can you think of a merging strategy?

Merging Two Sorted Arrays

```
void merge_arrays(Item merged[], Item a[], int N, Item b[], int M)
   {
   int i = 0, j = 0, k;

   for (k = 0; k < M + N; k++)
        {
        if (i == N)
           { merged[k] = b[j++]; continue; }

        if (j == M)
           { merged[k] = a[i++]; continue; }

        if (isLess(b[j], a[i]))
           { merged[k] = b[j++]; }
        else
           { merged[k] = a[i++]; }
        }
   }
```

“In-Place” Merge

- For sorting, we would like to:
  - Starting with sorted halves
    - `a[start … m]`
    - `a[m + 1 … end]`
  - Generate a sorted stretch
    - `a[start … end]`
- We would like an in-place merge, but…
  - A true “in-place” merge is quite complicated

Abstract In-Place Merge

- For caller, performs like in-place merge

- Creates copies two sub-arrays
- Replaces contents with merged results

C Code: Abstract In-place Merge (First Attempt)

```
void merge(Item a[], int start, int m, int stop)
  {
  static Item extra1[MAX_N];
  static Item extra2[MAX_N];

  for (int i = start; i <= m; i++)
      extra1[i - start] = a[i];

   for (int i = m + 1; i <= stop; i++)
      extra2[i – m - 1] = a[i];

  merge_arrays(a + start, extra1, m – start + 1,
                          extra2, stop – m);
  }
```

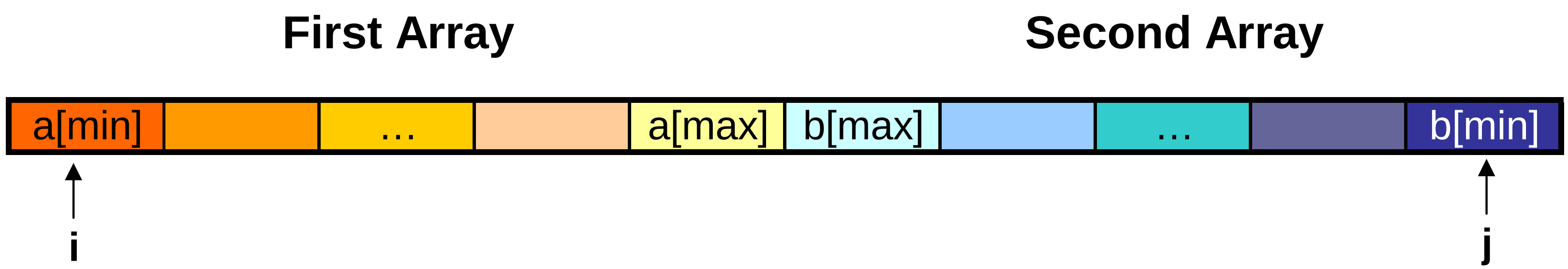
C Code: Abstract In-place Merge (Second Attempt)

```
void merge(Item a[], int start, int m, int stop)
  {
   static Item extra[MAX_N];

  for (int i = start; i <= stop; i++)
      extra[i] = a[i];

  for (int i = start, k = start, j = m + 1; k <= stop; k++)
      if (j<=stop && isLess(extra[j], extra[i]) || i>m)
         a[k] = extra[j++];
      else
         a[k] = extra[i++];
  }
```

Avoiding End-of-Input Check



At each point, compare elements i and j.

Then select the smallest element.

Move i or j towards the middle, as appropriate.

C Code: Abstract In-place Merge (Third Attempt!)

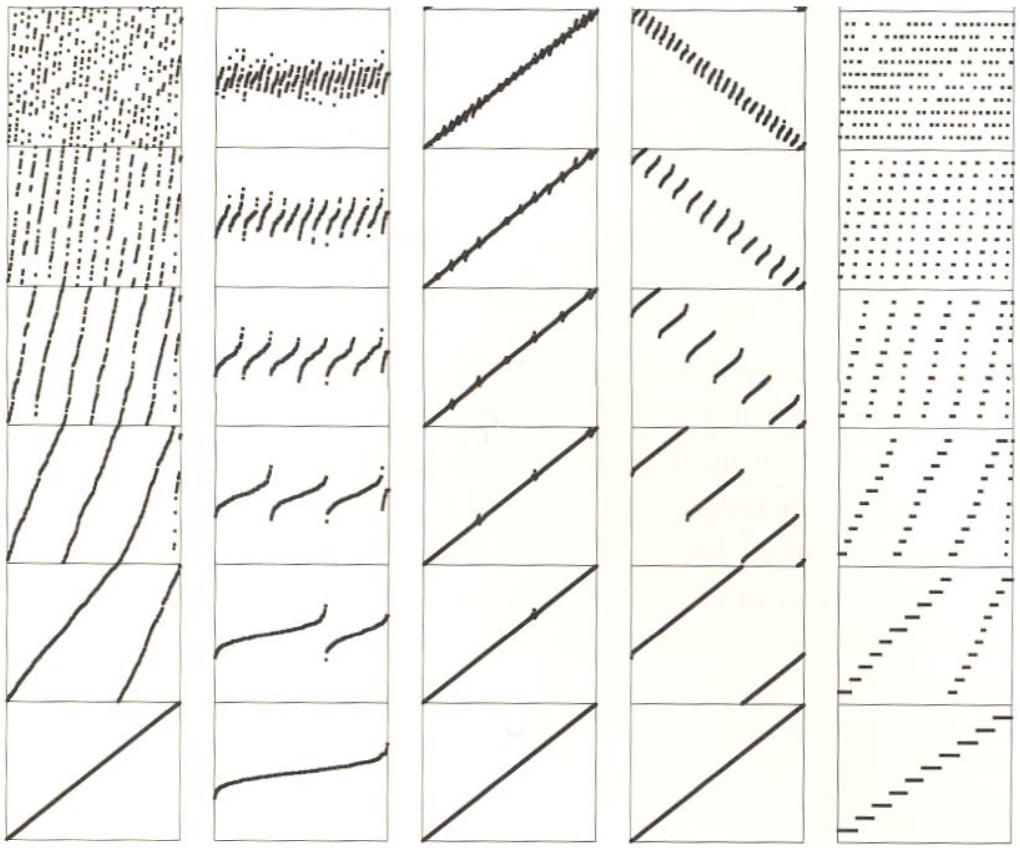
```
void merge(Item a[], int start, int m, int stop)
   {
   int i, j, k;

   for (int i = start; i <= m; i++)
       extra[i] = a[i];

   for (int j = m + 1; j <= stop; j++)
       extra[m + 1 + stop – j] = a[j];

   for (int i = start, k = start, j = stop; k <= stop; k++)
       if (isLess(extra[j], extra[i]))
          a[k] = extra[j--];
       else
          a[k] = extra[i++];
   }
```

Merge Sort in Action



Merge Sort Notes

- Order *N log N*
  - Number of comparisons independent of data
  - Exactly *log N* rounds
  - Each requires *N* comparisons
- Merge sort is stable
- Insertion sort for small arrays is helpful

Sedgewick's Timings (secs)

N	QuickSort	MergeSort	MergeSort*
100,000	24	53	43
200,000	52	111	92
400,000	109	237	198
800,000	241	524	426

Array of floating point numbers; * using insertion for small arrays

Non-Recursive Merge Sort

- First sort all sub-arrays of 1 element
- Perform successive merges
  - Merge results into sub-arrays of 2 elements
  - Merge results into sub-arrays of 4 elements
  - …

Bottom-Up Merge Sort

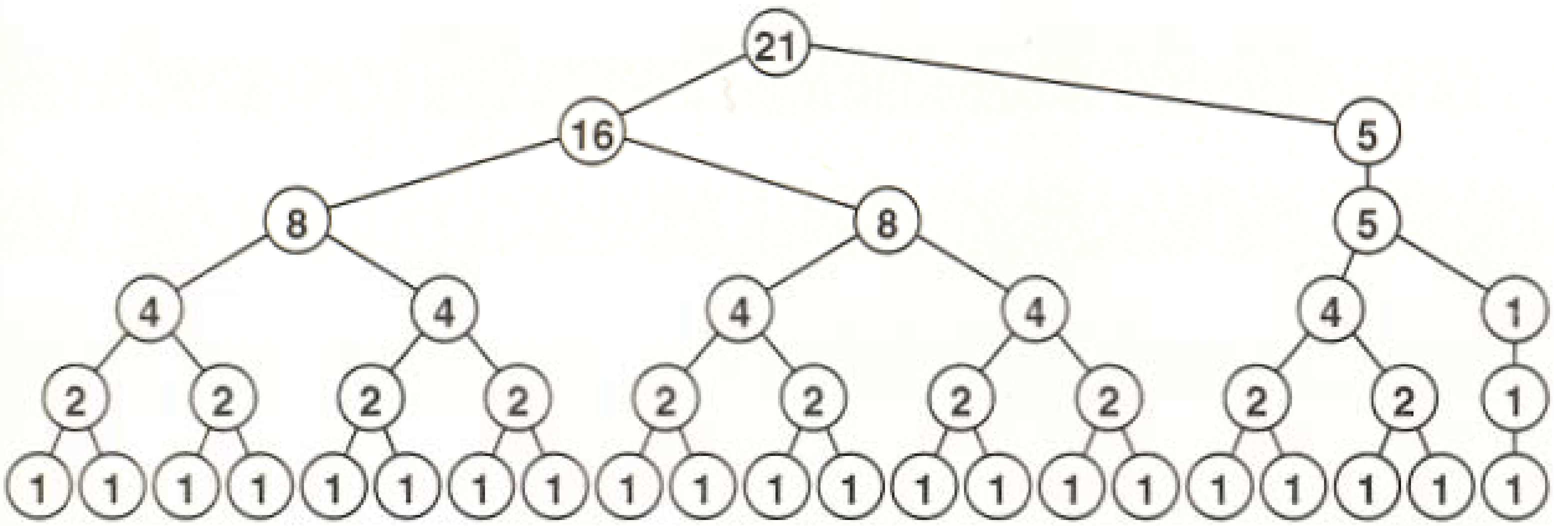
```
Item min(Item a, Item b)
  { return isLess(a,b) ? a : b; }

void mergesort(Item a[], int start, int stop)
   {
   int i, m;

   for (m = 1; m < stop – start; m += m)
     for (i = start; i < stop - m; i += m + m)
       {
       int from = i;
       int mid = i + m – 1;
       int to = min(i + m + m – 1, stop);

       merge(a, from, mid, to);
       }
   }
```

Merging Pattern for N = 21



Sedgewick's Timings (secs)

N	QuickSort	Top-Down MergeSort	Bottom-Up MergeSort
100,000	24	53	59
200,000	52	111	127
400,000	109	237	267
800,000	241	524	568

Array of floating point numbers

Automatic Memory Allocation

- Defining large static arrays is not efficient
  - Often, program will run on smaller datasets and the arrays will just waste memory
- A better way is to allocate and free memory as needed
- Create a “wrapper” function that takes care of memory allocation and freeing

Merge Sort,
With Automatic Memory Allocation

```
Item * extra;

void sort(Item a[], int start, int stop)
   {
   // Nothing to do with less than one element
   if (stop <= start) return;

   // Allocate the required extra storage
   extra = malloc(sizeof(Item) * (stop – start + 1));

   // Merge and sort the data
   mergesort(a, start, stop);

   // Free memory once we are done with it
   free(extra);
   }
```

Today …

- Contrasting approaches to divide and conquer
  - Quick Sort
  - Merge Sort

- Abstraction in functions
  - Some functions look simple for caller …
  - … but are more complex "under-the-hood"

- Unraveled Recursive Sorts

Sorting Summary

- Simple *$O(N^2)$* sorts for very small datasets
  - Insertion, Selection and Bubblesort
- Improved, but more complex sort
  - Shell sort
- Very efficient *N log N* sorts
  - Quick Sort (requires no additional storage)
  - Merge Sort (requires a bit of additional memory)

Sorting Indexes

- Generating an index is an alternative to sorting the raw data
- Allows us to keep track of many different orders
- Can be faster when items are large
- How it works:
  - Leaves the array containing the data unchanged
  - Generates an array where position i records position of the i^{th} smallest item in the original data

Example:
Indexing with Insertion Sort

```
void makeIndex(int index[], Item a[], int start, int stop)
  {
  for (int i = start; i <= stop; i++)
      index[i] = i;

  for (int i = start + 1; i <= stop; i++)
    for (int j = i; j > start; j--)
       if (isLess(a[index[j]], a[index[j-1]]))
          Exchange(index[j-1], index[j])
       else
          break;
  }
```

Next Lecture: An Alternative to Sorting

- We'll see how to organize data so that it can be searched …
- And so the complexity of searching and organizing the data is less than $N \log N$
- Cost: Doing this will require additional memory

Recommended Reading

- For QuickSort
  - Sedgewick, Chapter 7
  - Hoare (1962) *Computer Journal* **5:**10-15.
- For MergeSort
  - Sedgewick, Chapter 8